Task 1. Kinetic energy of the car

a. Solution

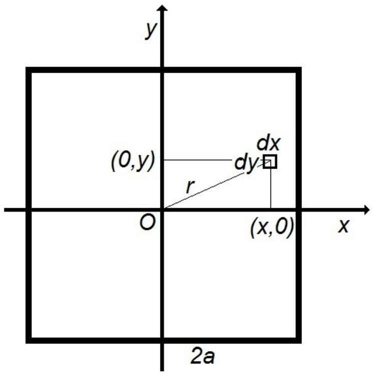

Figura 1 Calculul momentului de inerţie

The density of wheel material (if the length of wheel is L) is e

$$\rho = \frac{m}{4a^2 \cdot L} \quad (1)$$

Considering the elementary prism having the sides (L, dx, dy), the elementary with respect of wheel axis is

$$dJ = \rho \iint r^2 \cdot L \cdot dx \cdot dy \quad (2)$$

with

$$r^2 = x^2 + y^2 \quad (3)$$

Because

$$\int_{-a}^{a} \left(x^2 + y^2\right) dy = \left(x^2 \cdot y + \frac{y^3}{3}\right)\Bigg|_{-a}^{a} = 2a \cdot x^2 + 2\frac{a^3}{3} \quad (4)$$

results

$$\int_{-a}^{a} \left(2a \cdot x^2 + 2\frac{a^3}{3}\right) dx = \left(2a \cdot \frac{x^3}{3} + 2\frac{a^3}{3} \cdot x\right)\Bigg|_{-a}^{a} = \frac{4a^4}{3} + \frac{4a^4}{3} = \frac{8a^4}{3} \quad (5)$$

The moment of inertia of a wheel is

$$J = \frac{m}{4a^2 L} \cdot L \cdot \int_{-a}^{a}\left(\int_{-a}^{a}\left(x^2 + y^2\right) dy\right) dx = \frac{m}{4a^2}\int_{-a}^{a}\left(\int_{-a}^{a}\left(x^2 + y^2\right) dy\right) dx = \frac{m}{4a^2} \cdot \frac{8a^4}{3} = \frac{2a^2 \cdot m}{3} \quad (6)$$

b. Solution

When the wheel is “on top” or “on valley” the translation speed of axles of wheel (that is the translation speed of the car) $v(t)$ - and the radius of gyration r $r(t)$ are perpendicular so that

$$v(t) = r(t) \cdot \omega(t) \quad (7)$$

where $\omega(t)$ is the angular velocity of the wheel.

When the motion starts - in the top of a bump, the wheel having the radius of gyration a, the instantaneous angular speed of the wheel, ω_T, will be

$$\omega_T = \frac{v_0}{a} \qquad (8)$$

Translational kinetical energy T_t is

$$T_t = \frac{M + 2m}{2} v^2 \qquad (9)$$

and rotational kinetical energy T_r in the same point is

$$T_r = 2\frac{J}{2}\omega^2 = J\omega^2 = \frac{2a^2 \cdot m}{3} \cdot \omega^2 . \qquad (10)$$

The total kinetical energy T of the car is

$$T = T_t + T_r = \frac{M + 2m}{2} v^2 + \frac{2a^2 \cdot m}{3} \cdot \omega^2 \qquad (11)$$

At the beginning $v = v_0$ and $\omega_T = \frac{v_0}{a}$

$$T = v_0^2 \cdot \left(\frac{M}{2} + m + \frac{2m}{3} \right) = v_0^2 \cdot \frac{3M + 10m}{6} \qquad (12)$$

In the valley, the translation speed v_v and angular speed satisfy

$$\omega_V = \frac{v_v}{a\sqrt{2}}; \; v_v = \omega_V \cdot a\sqrt{2} \qquad (13)$$

In the valley, the total kinetical energy is

$$T = \left(\frac{(M + 2m) \cdot 2a^2}{2} + \frac{2a^2 \cdot m}{3} \right) \cdot \omega_v^2 = \left(M + 2m + \frac{2 \cdot m}{3} \right) \cdot \omega_v^2 \cdot a^2 = \omega_v^2 \cdot a^2 \cdot \frac{3M + 8m}{3} \qquad (14)$$

The potential energy of the car do not change (because the center of mass remains at the same height as stated in statement). Because no external action is allowed, the total kinetical energy is conserved that means (using (12) and (14))

$$\left(\frac{3M + 8 \cdot m}{3} \right) \cdot \omega_v^2 \cdot a^2 = v_0^2 \cdot \left(\frac{3M + 10m}{6} \right) \qquad (15)$$

and

$$\omega_v = \frac{v_0}{a} \cdot \sqrt{\frac{3M + 10m}{6M + 16m}} \qquad (16)$$

Analyzing .the relation above one may see that

i. If $M >> m$, from (16) results

$$\omega_v = \frac{v_0}{a\sqrt{2}} \; ; \; v_v = v_0 \qquad (17)$$

The translation speed remains constant if the wheels are unimportant

ii. If $M << m$, from (16) results

$$\omega_v = \frac{v_0}{a} \cdot \sqrt{\frac{5}{8}} \qquad (18)$$

iii. If $M = m$, from (16) results

$$\omega_v = \frac{v_0}{a} \cdot \sqrt{\frac{13}{22}} \qquad (19)$$

Task 2. Road's shape

a. If the weight would have a moment relative to the contact point, it would appear a transfer of energy from potential to kinetical or vice versa that being against statement hypothesis.

b. In the image in the figure 2 is presented the motion of contact side of the wheel (in cross section) over the road's shape – the curve passing through x_s, T, x_d.

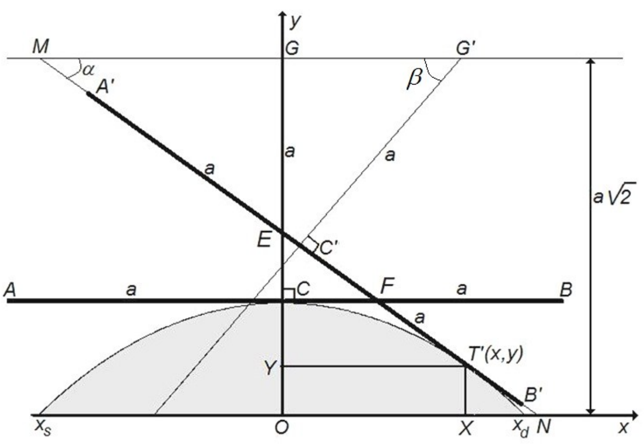

Figure 2. Cross section view of the motion of the wheel on an elementary bump of the road. The vertical passing through the center of mass of the wheel passes also through the contact point. The wheel never glides.

Using the hypothesis

$$tg\,\alpha = -y'(x) \qquad (20)$$

Because

$$\cos\alpha = \frac{1}{\sqrt{1 + tg^2\alpha}} \qquad (21)$$

results that

$$\cos\alpha = \frac{1}{\sqrt{1 + y'^2}} \qquad (22)$$

so that

$$G'T' = \frac{a}{\cos\alpha} = a \cdot \sqrt{1 + y'^2} \qquad (23)$$

Co linearity of G' of T' is

$$y + a \cdot \sqrt{1 + y'^2} = a\sqrt{2} \qquad (24)$$

that is

$$y' = \sqrt{\left(\sqrt{2} - \frac{y}{a}\right)^2 - 1} \qquad (25)$$

and

$$\frac{dy}{\sqrt{\left(\sqrt{2} - \frac{y}{a}\right)^2 - 1}} = \pm dx \qquad (26)$$

Integrating (26) one obtain that

$$\pm \frac{x}{a} + \wp = ln\left|\left(\sqrt{2} - y/a\right) + \sqrt{\left(\sqrt{2} - y/a\right)^2 - 1}\right| \qquad (27)$$

Because for

$$x = 0\,,\ y = a \cdot \left(\sqrt{2} - 1\right), \qquad (28)$$

The constant in integral is,

$$\wp = 0 \qquad (29)$$

and because

$$y < a\left(\sqrt{2} - 1\right) \qquad (30)$$

expression (27) can be written as

$$\left(\sqrt{2} - y/a\right) + \sqrt{\left(\sqrt{2} - y/a\right)^2 - 1} = e^{\pm\frac{x}{a}} \qquad (31)$$

$$\left(e^{\pm\frac{x}{a}} - \left(\sqrt{2} - y/a\right)\right)^2 = \left(\sqrt{2} - y/a\right)^2 - 1 \qquad (32)$$

$$\frac{e^{\pm\frac{2x}{a}} + 1}{e^{\pm\frac{x}{a}}} = 2\left(\sqrt{2} - y/a\right); \frac{e^{\pm\frac{2x}{a}} + e^{\mp\frac{2x}{a}}}{2} = \sqrt{2} - \frac{y}{a} \qquad (33)$$

or

$$y = a\left(\sqrt{2} - ch\,(x/a)\right) \qquad (34)$$

The graph of function in (34) is presented in figure (3).

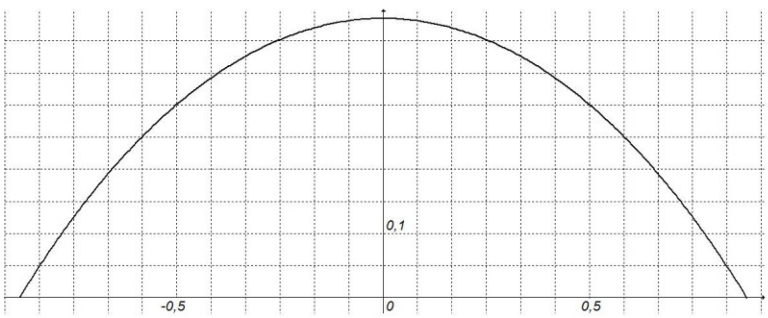

Figure 3 Elementary relief of the road

c.

from (31) for $y = 0$ results

$$\pm x = a \cdot \ln\left(\sqrt{2} + 1\right) \qquad (35)$$

The ends of an elementary relief are

$$\begin{cases} x_s = -a \cdot \ln\left(\sqrt{2} + 1\right) \\ x_d = \ \ a \cdot \ln\left(\sqrt{2} + 1\right) \end{cases} \qquad (36)$$

So that the horizontal extension of the bump $2d$, is

$$2d = \left|x_d - x_s\right| = 2a \cdot \ln\left(\sqrt{2} + 1\right) \qquad (37)$$

d.

Because

$$2a \cdot \ln\left(\sqrt{2} + 1\right) \cong 1{,}76\, a < 2\sqrt{2}a \qquad (38)$$

The two wheels cannot rotate if they are over two neighbor bumps. The minimal distance between axles $d_{\min}$ must be

$$d_{\min} = 4a \cdot \ln\left(\sqrt{2} + 1\right) \qquad (39)$$

e.

Using the expression of road's shape results that,

$$y' = -sh\,(x/a) \qquad (40)$$

Consequently

$$\begin{cases} y'(x_d) = -1 \\ y'(x_s) = 1 \end{cases} \qquad (41)$$

The angle between the tangents to relief in valley , $\frac{\pi}{2}$, is exactly appropriate for a square wheel. No hexagonal prism can fit the road.

Task3. Accident

The speed of center of mass obey to relation

$$v_{G'} = \frac{d}{dt}(GG') = \frac{dx}{dt} \qquad (42)$$

During the motion the gyration radius that initially is in position GC reach the position $G'C'$ rotating with the angle

$$\beta = \frac{\pi}{2} - \alpha \qquad (43)$$

Because, as stated in statement

$$\alpha = arctg\,(-y'(x)) \qquad (44)$$

The angular speed of the wheel is

$$\omega = \frac{d\beta}{dt} = -\frac{d\alpha}{dt} = -\frac{d}{dt}\left[arctg\,(-y')\right] = \frac{d}{dt}\left[arctg\,(y')\right] \qquad (45)$$

$$\omega = -\frac{d}{dt}\left(arctg\ \left(sh\ \left(x/a\right)\right)\right) = -\frac{\left(ch\ \left(x/a\right)\right)}{1+\left(sh\ \left(x/a\right)\right)^2}\cdot\frac{1}{a}\cdot\frac{dx}{dt} \qquad (46)$$

$$\omega = -\frac{1}{a\cdot ch\ \left(x/a\right)}\cdot\frac{dx}{dt} \qquad (47)$$

When the wheel evolves from top to valley

$$0 < x < d = a\cdot\ln\left(\sqrt{2}-1\right) \qquad (48)$$

And correspondingly

$$1 < ch\ \left(x/a\right) < \sqrt{2} \qquad (49)$$

Between the translational and angular speeds it is the relationship

$$\omega = -\frac{1}{a\cdot ch\ \left(x/a\right)}\cdot v_{G'} \qquad (50)$$

During the motion the potential energy remains constant..

The translational kinetical energy -are (9) $T_t = \left(M+2m\right)\cdot v_{G'}^2/2$ and the rotational kinetical energy are (10) - $T_r = J\cdot\omega^2$ a. So,

$$T = T_t + T_r = \frac{\left(M+2m\right)}{2}v_{G'}^2 + J\cdot\omega^2 \qquad (51)$$

$$T = \frac{\left(M+2m\right)}{2}v_{G'}^2 + \frac{2a^2\cdot m}{3}\cdot\left(\frac{1}{a\cdot ch\left(x/a\right)}\cdot v_{G'}\right)^2 \qquad (52)$$

$$T = v_{G'}^2\cdot\left(\frac{\left(M+2m\right)}{2}+\frac{2}{3}m\cdot\frac{1}{ch^2\left(x/a\right)}\right) \qquad (53)$$

At the beginning $x = 0$

$$v_{G'}(0) = v_0\ ;\ ch\ (x/a) = 1 \qquad (54)$$

$$T = v_0^2\cdot\left(\frac{M}{2}+\frac{5m}{3}\right) \qquad (55)$$

that is in agreement with (12)

Generally, the dependence of speed on position has the

$$v_{G'} = \frac{v_0\cdot\sqrt{\left(\frac{M}{2}+\frac{5m}{3}\right)}}{\sqrt{\frac{\left(M+2m\right)}{2}+\frac{2}{3}m\cdot\frac{1}{ch^2\left(x/a\right)}}} \qquad (56)$$

When moving from top to valley, $0 < x < a\cdot\ln\left(\sqrt{2}+1\right)$ and correspondingly $1 < \cosh\frac{x}{a} < \sqrt{2}$.

In considered domain $ch\frac{x}{a}$ monotonically increases and his extremes values appears at the ends.

$$\frac{1}{2} < \frac{1}{ch^2(x/a)} < 1 \qquad (57)$$

The extremes values are initial speed v_0 and the speed in valley

$$v_{G'}(d) = \frac{v_0 \cdot \sqrt{3M + 10m}}{\sqrt{3M + 8m}} > v_0 \qquad (58)$$

Relation (58) is the answer for question d.

d. The heat delivered by collision DO NOT depends on the position of the collision. Whatever the point of collision will be the constant total kinetical energy is totally transformed in heat..

e. The heat produced Q equals the total kinetical energy

$$Q = v_0^2 \cdot \left(\frac{3M + 10m}{6}\right) \qquad (59)$$

The relation (59) is the answer to the question e.

Compton scattering - Solution

Task 1 - First collision

1.a. Figure 1 presents the situation before the first scattering of the photon.

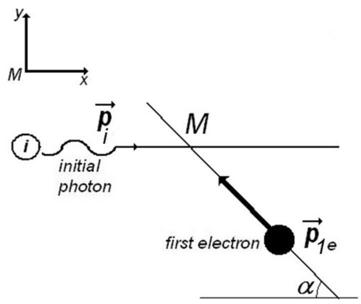

Figure 1 Photon and electron before the first collision. The collision occurs in the point M

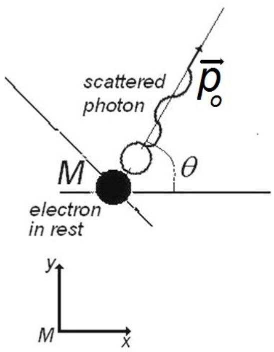

Figure 2. Photon and electron after the first collision. The collision took place in point M

1.b. The angle between the directions of the electron and photon moments before collision is denoted by α

To characterize the initial photon we will use its moment $\vec{p}_i$ and its energy E_i

$$\begin{cases} \left|\vec{p}_i\right| = \dfrac{h}{\lambda_i} = \dfrac{h \cdot f_i}{c} \\ E_i = h \cdot f_i \end{cases} \qquad (1)$$

$$f_i = \frac{c}{\lambda_i} \qquad (2)$$

is the frequency of the initial photon.

For the initial free electron in motion scenario, the moment $\vec{p}_{1e}$ and the energy E_{oe} are

$$\begin{cases} \vec{p}_{1e} = m \cdot \vec{v}_{1e} = \dfrac{m_0 \cdot \vec{v}_{1e}}{\sqrt{1-\beta^2}} \\ E_{1e} = m \cdot c^2 = \dfrac{m_0 \cdot c^2}{\sqrt{1-\beta^2}} \end{cases} \qquad (3)$$

where m_0 is the rest mass of the electron and m is the mass of the moving electron. As usual $\beta = \dfrac{v_{1e}}{c}$. De Broglie wavelength of the first electron is

$$\lambda_{1e} = \frac{h}{p_{1e}} = \frac{h \cdot}{m_0 \cdot v_{1e}} \sqrt{1-\beta^2} \qquad (4)$$

1.c. The situation after the scattering of photon is described in the figure 2.
To characterize the scattered photon we will use its moment $\vec{p}_0$ and its energy E_0

$$\begin{cases} \left|\vec{p}_o\right| = \dfrac{h}{\lambda_o} = \dfrac{h \cdot f_o}{c} \\ E_o = h \cdot f_o \end{cases} \qquad (5)$$

where

$$f_o = \frac{c}{\lambda_0} \qquad (6)$$

is the frequency of the scattered photon.
The magnitude of the moment of the electron (that remains at rest) after the scattering is zero; its energy is E_{0e}. The mass of the electron after collision is m_0 - the mass of electron at rest state. So,

$$E_{0e} = m_0 \cdot c^2 \qquad (7)$$

To determine the moment of the first moving electron, one can write the principles of conservation of moments and energy. That is

$$\vec{p}_i + \vec{p}_{1e} = \vec{p}_0 \qquad (8)$$

and

$$E_i + E_{1e} = E_0 + E_{0e} \qquad (9)$$

Using the referential in figures 1 and 2, the conservation of the moment in collision along the Ox direction is written as

$$\frac{h \cdot f_i}{c} + m \cdot v_{1e} \cdot \cos\alpha = \frac{h \cdot f_0}{c} \cos\theta \qquad (10)$$

and the conservation of moment along Oy direction is

$$m \cdot v_{1e} \cdot \sin\alpha = \frac{h \cdot f_0}{c} \sin\theta \qquad (11)$$

To eliminate α, the last two equations must be written again as

$$\begin{cases} \left(m \cdot v_{1e} \cdot \cos\alpha\right)^2 = \dfrac{h^2 \cdot}{c^2}\left(f_0 \cdot \cos\theta - f_i\right)^2 \\ \left(m \cdot v_{1e} \cdot \sin\alpha\right)^2 = \left(\dfrac{h \cdot f_0}{c} \sin\theta\right)^2 \end{cases} \qquad (12)$$

and then added.
The result is

$$m^2 \cdot v_{1e}^2 = \frac{h^2 \cdot}{c^2}\left(f_0^2 + f_i^2 - 2 f_0 \cdot f_i \cdot \cos\theta\right) \qquad (13)$$

or

$$\frac{m_0^2 \cdot c^2}{1-\left(\dfrac{v_{1e}}{c}\right)^2} \cdot v_{1e}^2 = h^2 \cdot \left(f_0^2 + f_i^2 - 2 f_0 \cdot f_i \cdot \cos\theta\right) \qquad (14)$$

The conservation of energy (9) can be written again as

$$m \cdot c^2 + h \cdot f_i = m_0 \cdot c^2 + h \cdot f_0 \qquad (15)$$

or

$$\frac{m_0 \cdot c^2}{\sqrt{1-\left(\frac{v_{1e}}{c}\right)^2}} = m_0 \cdot c^2 + h \cdot (f_0 - f_i) \qquad (16)$$

Squaring the last relation results in the following

$$\frac{m_0^2 \cdot c^4}{1-\left(\frac{v_{1e}}{c}\right)^2} = m_0^2 \cdot c^4 + h^2 \cdot (f_0 - f_i)^2 + 2m_0 \cdot h \cdot c^2 \cdot (f_0 - f_i) \qquad (17)$$

Subtracting (14) from (17) the result is

$$2m_0 \cdot c^2 \cdot h \cdot (f_0 - f_i) + 2h^2 \cdot f_i \cdot f_0 \cdot \cos\theta - 2h^2 \cdot f_i \cdot f_0 = 0 \qquad (18)$$

or

$$\frac{h}{m_0 \cdot c}(1-\cos\theta) = \frac{c}{f_i} - \frac{c}{f_0} \qquad (19)$$

Using

$$\Lambda = \frac{h}{m_0 \cdot c} \qquad (20)$$

the relation (19) becomes

$$\Lambda \cdot (1-\cos\theta) = \lambda_i - \lambda_0 \qquad (21)$$

The wavelength of scattered photon is

$$\lambda_0 = \lambda_i - \Lambda \cdot (1-\cos\theta) \qquad (22)$$

Equation (22) show that the scattered photon wavelength is shorter than the wavelength of the initial photon. Consequently the energy of the scattered photon is greater that the energy of the initial photon.

$$\begin{cases} \lambda_0 < \lambda_i \\ E_0 > E_i \end{cases} \qquad (23)$$

Task 2 - Second collision

2.a. Let's analyze now the second collision process that occurs at point N. To study that, let's consider a new referential having the Ox axis along the direction of the photon scattered after the first collision.

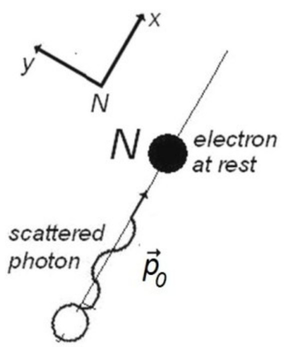

Figure 3 Photon and electron before the second collision. The collision occur at the point N

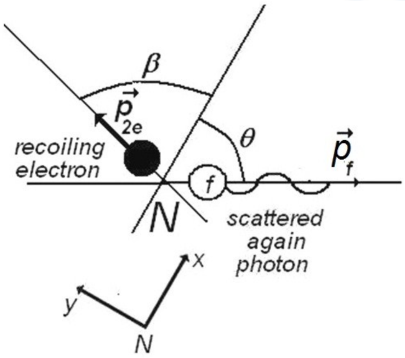

Figure 4 Photon and electron after the second collision. The collision took place at the point N. The scattering angle for electron is denoted by β.

Figure 3 presents the situation before the second collision and Figure 4 presents the situation after this scattering process.

2.b. The conservation law for moments in the scattering process gives

$$\begin{cases} \dfrac{h}{\lambda_0} = \dfrac{h}{\lambda_f} \cos\theta + m \cdot v_{2e} \cdot \cos\beta \\ \dfrac{h}{\lambda_f} \sin\theta - m \cdot v_{2e} \cdot \sin\beta = 0 \end{cases} \qquad (24)$$

To eliminate the unknown angle β, one must square and then add equations (24)

That is

$$\begin{cases} \left(\dfrac{h}{\lambda_0} - \dfrac{h}{\lambda_f} \cos\theta \right)^2 = (m \cdot v_{2e} \cdot \cos\beta)^2 \\ \left(\dfrac{h}{\lambda_f} \sin\theta \right)^2 = (m \cdot v_{2e} \cdot \sin\beta)^2 \end{cases} \qquad (25)$$

and

$$\left(\frac{h}{\lambda_f} \right)^2 + \left(\frac{h}{\lambda_0} \right)^2 - \frac{2 \cdot h^2}{\lambda_0 \cdot \lambda_f} \cos\theta = (m \cdot v_{2e})^2 \qquad (26)$$

or

$$\frac{h^2 \cdot c^2}{\lambda_f^2} + \frac{h^2 \cdot c^2}{\lambda_0^2} - \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f} \cos\theta = m^2 \cdot c^2 \cdot v_{2e}^2 \qquad (27)$$

The conservation law for energies in the second scattering process gives

$$\frac{h \cdot c}{\lambda_0} + m_0 \cdot c^2 = \frac{h \cdot c}{\lambda_f} + m \cdot c^2 \qquad (28)$$

or

$$h^2 \cdot c^2 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0} \right)^2 + m_0^2 \cdot c^4 + 2h \cdot c^3 \cdot m_0 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0} \right) = \frac{m_0^2 \cdot c^4}{1 - (v_{2e}/c)^2} \qquad (29)$$

Subtracting (27) from (29), one obtains

$$+ \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f} \cos\theta - \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f} + 2h \cdot c^3 \cdot m_0 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0} \right) = 0 \qquad (30)$$

that is

$$\begin{cases} \dfrac{h}{m_0 \cdot c} \cdot (1 - \cos\theta) = \lambda_f - \lambda_0 \\ \lambda_f - \lambda_0 = \Lambda \cdot (1 - \cos\theta) \end{cases} \quad (31)$$

Concluding

$$\begin{cases} \lambda_f > \lambda_0 \\ E_f < E_0 \end{cases} \quad (32)$$

Comparing (31) written as

$$\lambda_f = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \quad (33)$$

and (22) written as

$$\lambda_i = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \quad (34)$$

results in

$$\lambda_i = \lambda_f \quad (35)$$

2.c. Taking into account (26), the moment of the electron after the second collision is

$$p_{2e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\lambda_0^2} - \frac{2 \cdot \cos\theta}{\lambda_f \cdot \lambda_0}} \quad (36)$$

Or, considering (33), the moment can be rewritten as

$$p_{2e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1 - \cos\theta))^2} - \frac{2 \cdot \cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1 - \cos\theta))}} \quad (37)$$

The expression of the de Broglie wavelength of the second electron after scattering is

$$\lambda_{2e} = 1 \Bigg/ \left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1 - \cos\theta))^2} - \frac{2 \cdot \cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1 - \cos\theta))}} \right) \quad (38)$$

From (28) results

$$\frac{h \cdot c}{\lambda_0} = \frac{h \cdot c}{\lambda_f} + T_2 \quad (39)$$

that is

$$T_2 = h \cdot c \cdot \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f} \right) \quad (40)$$

Task 3 - Quantitative description of processes

3.a. The relation (13) can be written as

$$p_{1e} = h\sqrt{\frac{1}{\lambda_i^2} + \frac{1}{\lambda_0^2} - \frac{2 \cdot \cos\theta}{\lambda_i \cdot \lambda_0}} \quad (41)$$

or, considering (34) and (35)

$$p_{1e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1 - \cos\theta))^2} - \frac{2 \cdot \cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1 - \cos\theta))}} \quad (42)$$

so that the expression of the de Broglie wavelength of the first electron before scattering is

$$\lambda_{1e} = 1 \Bigg/ \left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1 - \cos\theta))^2} - \frac{2 \cdot \cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1 - \cos\theta))}} \right) \quad (43)$$

In the condition of the problem (43) becomes

$$\lambda_{1e} = 1 \Bigg/ \left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \frac{\Lambda}{2}\right)^2} - \frac{1}{\lambda_f \cdot \left(\lambda_f - \frac{\Lambda}{2}\right)}} \right) \qquad (44)$$

Because the value of λ_f and Λ are known as

$$\begin{cases} \lambda_f = \lambda_i = 1{,}25 \times 10^{-10}\ m \\ \Lambda = \dfrac{6{,}6 \times 10^{-34}}{9{,}1 \times 10^{-31} \cdot 3{,}0 \times 10^{8}}\ m = 2{,}41 \times 10^{-12}\ m = 0{,}02 \times 10^{-10}\ m \end{cases} \qquad (45)$$

From (33) results that

$$\lambda_0 = \lambda_f - \Lambda \cdot (1 - \cos\theta) = \lambda_f - \frac{\Lambda}{2} \qquad (46)$$

or

$$\lambda_0 = 1{,}24 \times 10^{-10}\ m \qquad (47)$$

The numerical value of the de Broglie wavelength of the first electron before scattering is

$$\lambda_{1e} = 10^{-10} \Bigg/ \left(\sqrt{\frac{1}{1{,}25^2} + \frac{1}{1{,}24^2} - \frac{1}{1{,}25 \times 1{,}24}} \right) m \cong 1{,}25 \times 10^{-10}\ m \qquad (48)$$

Evidently the de Broglie wavelength λ_{2e} for the second electron after the scattering is the same

$$\lambda_{1e} = \lambda_{2e} \qquad (49)$$

3.b. Because $\lambda_f = \lambda_i$ the energy of the first photon before the first scattering has the following expression

$$E_i = \frac{h \cdot c}{\lambda_i} = \frac{h \cdot c}{\lambda_f} \qquad (50)$$

Its numerical value is

$$E_i = \frac{h \cdot c}{\lambda_i} = \frac{6{,}6 \times 10^{-34} \cdot 3{,}0 \times 10^{8}}{1{,}25 \times 10^{-10}} \cong 1{,}58 \times 10^{-15}\ J = 9{,}9\,KeV \qquad (51)$$

The frequency f_i of the first photon before the first scattering has the following expression

$$f_i = \frac{c}{\lambda_i} = \frac{c}{\lambda_f} \qquad (52)$$

Its numerical value is

$$f_i = \frac{3 \times 10^{8}}{1{,}25 \times 10^{-10}} \cong 2{,}4 \times 10^{18}\ Hz \qquad (53)$$

3.c. From (40) written as

$$T_2 = T_1 = m \cdot c^2 - m_0 \cdot c^2 = h \cdot c \cdot \left(\frac{1}{\lambda_0} - \frac{1}{\lambda_f} \right) \qquad (54)$$

results

$$m = m_0 + \frac{h}{c} \cdot \left(\frac{1}{\lambda_f - (\Lambda/2)} - \frac{1}{\lambda_f} \right) \cong m_0 + \frac{h}{c} \cdot \frac{\Lambda}{2\lambda_f^2} = m_0 \cdot \left[1 + \frac{1}{2} \left(\frac{h}{m_0 \cdot c \cdot \lambda_f} \right)^2 \right] \qquad (55)$$

Its numerical value is

$$m \cong m_0 \qquad (56)$$

Consequently, the speed of the second electron after the collision v_{2e} has the following expression

$$v_{2e} = \frac{h}{\lambda_f \cdot m} \approx \frac{h}{\lambda_f \cdot m_0} \qquad (57)$$

Its numerical value is

$$v_{2e} \approx 5{,}8 \times 10^6 \, m/s = 0{,}019 \cdot c \qquad (58)$$

3.d. As results from (46)

$$\lambda_i - \lambda_0 = \lambda_f - \lambda_0 = \frac{\Lambda}{2} \qquad (59)$$

The numerical value of the variation of wavelength is

$$\frac{\Lambda}{2} \cong 0{,}01 \times 10^{-10} \, m \qquad (60)$$

© Solution proposed by
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Task 1. The link between image and the actual position

The image appears in the camera because of rays of light r*eaching* in the same time and no because of light rays *leaving* simultaneously. For this reason, there is difference between the image of a point on the line of beacons and the current position of this point at the time the image is formed. A beacon which image is in the position denoted x_i if the light coming from him has left him before the moment when the image is taken with the time T

$$T = \frac{\sqrt{d^2 + x_i^2}}{c} \qquad (1)$$

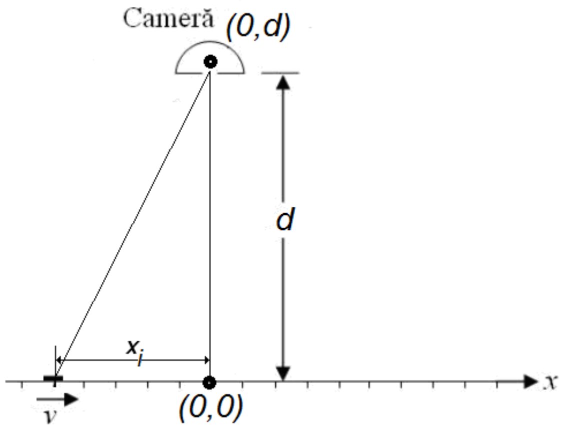

Figure 0.1

In the time it takes light to reach the camera, a distance is traveled by the beacon so that "current position" of the beacon (when the image is taken) is

$$x = x_i + v \cdot T = x_i + \frac{v}{c}\sqrt{d^2 + x_i^2} = x_i + \beta \cdot \sqrt{d^2 + x_i^2} \qquad (2)^*$$

The beacon situated at x_i in the image, appears on image when its actually position is x.

(Relation (2) represents the answer to the question ***1.a****)*

Squaring (2) results

$$\begin{cases} x^2 - 2x \cdot x_i + x_i^2 = \beta^2 \cdot d^2 + \beta^2 \cdot x_i^2 \\ (1-\beta^2)x_i^2 - 2x \cdot x_i + (x^2 - \beta^2 \cdot d^2) = 0 \\ (1-\beta^2)x_i^2 - 2x \cdot x_i + (x^2 - d^2 + (1-\beta^2)\cdot d^2) = 0 \\ \frac{x_i^2}{\gamma^2} - 2x \cdot x_i + \left(x^2 - d^2 + \frac{d^2}{\gamma^2}\right) = 0 \end{cases} \qquad (3)$$

Correspondingly,

$$\begin{cases} x_i = \gamma^2\left(x \pm \sqrt{x^2 - \frac{1}{\gamma^2}(x^2 - \beta^2 \cdot d^2)}\right) \\ x_i = \gamma^2\left(x \pm \frac{1}{\gamma}\sqrt{\beta^2 \cdot d^2 - (1-\gamma^2)\cdot x^2}\right) \end{cases} \qquad (4)$$

Because

$$1-\gamma^2 = 1-\frac{1}{1-\beta^2} = \frac{-\beta^2}{1-\beta^2} = -\beta^2 \cdot \gamma^2 \qquad (5)$$

from the last relation in (5) results

$$\begin{cases} x_i = \gamma^2 \left(x \pm \frac{\beta}{\gamma} \sqrt{d^2 + \gamma^2 \cdot x^2} \right) \\ x_i = \gamma \left(\gamma \cdot x \pm \beta \sqrt{d^2 + \gamma^2 \cdot x^2} \right) \end{cases} \qquad (6)$$

Because the beacons approaching at origin of the axis, natural selection of sign în expression above is

$$x_i = \gamma \left(\gamma \cdot x - \beta \cdot \sqrt{d^2 + \gamma^2 \cdot x^2} \right) \qquad (7)^*$$

Beacons whose image appears in the camera image in the position x_i is at position x when the image is formed

*(Relation (7) represents the answer to the question **1.b**)*

Task 2. Apparent length of the beacons line

Because the beacons are moving with the speed $\vec{v}$, the length of beacons line is (because of Lorentz contraction) -. L/γ .

In the moment when the camera takes the image of beacons line having the third beacon in the position x_0, the beacons situate at the ends of line have respectively the positions

$$x_{fa\,ţa} = x_+ = x_0 + \frac{L}{2\gamma} \qquad (8)$$

and

$$x_{spate} = x_- = x_0 - \frac{L}{2\gamma} \qquad (9)$$

The images of the ends of beacons line $x_{i,\pm}$ will be formed by the camera „as is” thei are in the positions

$$x_{i,\pm} = \gamma \left(\gamma \cdot x_0 \pm \frac{L}{2} \right) - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 \pm \frac{L}{2} \right)^2} \qquad (10)$$

In the relation above was taken into consideration the expressions of positions of ends of beacons line and the relation (8).

The apparent length of beacons line L_i, represents the difference of apparent positions of the ends of beacons line that is

$$L_i(x_0) = x_{i,+} - x_{i,-} \qquad (11)$$

The answer to the question (2.a) is (12)

$$L_i(x_0) = \gamma \cdot L - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2} \right)^2} + \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 - \frac{L}{2} \right)^2} \qquad (12)^*$$

The expression below is a second degree dependence on x_0,

$$F_1 = \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2 + d^2 \qquad (13)$$

And has a minimum, d^2, for

$$x_0 = -\frac{L}{2\gamma} \qquad (14)$$

For $x_0,\ x_0 \geq -\frac{L}{2\gamma}$ the F_1 monotonically increases.

Correspondingly, the expression

$$E_1 = -\beta \cdot \gamma \sqrt{F_1} \qquad (15)$$

Monotonically increases to $x_0 = -\frac{L}{2\gamma}$, and then monotonically deceases and has a maximum

$$E_{1,\max} = E_1\left(x_0 = -\frac{L}{2\gamma}\right) = -\beta\,\gamma\, d \qquad (16)$$

For

$$x_0 = \frac{L}{2\gamma} \qquad (17)$$

$$E_1\left(x_0 = \frac{L}{2\gamma}\right) = -\beta \cdot \gamma \sqrt{d^2 + L^2} \qquad (18)$$

Graphically, the dependence $E_1(x_0)$ looks like in ***Figure .3***

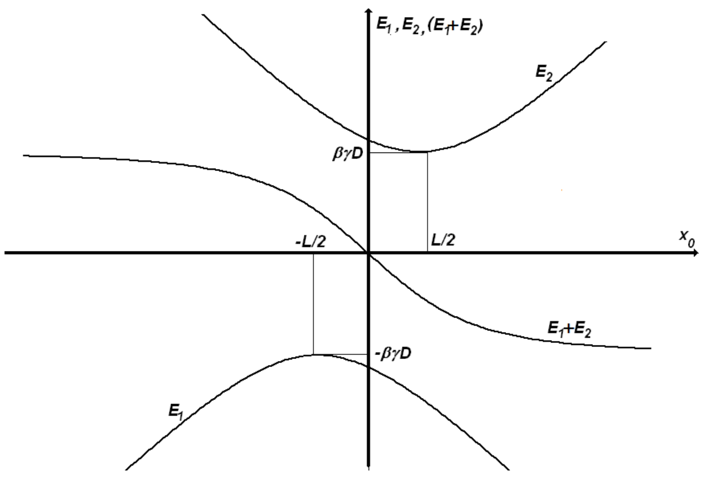

Figure 2 Reprezentarea grafică a funcţiilor E_1, E_2 şi $E_1 + E_2$

Analogous, as function on x_0 the polynomial expression

$$F_2 = \left(\gamma \cdot x_0 - \frac{L}{2}\right)^2 + d^2 \qquad (19)$$

has as minimum d^2 for

$$x_0 = \frac{L}{2\gamma} \qquad (20)$$

If $x_0,\ x_0 \geq \frac{L}{2\gamma}\quad F_2$ monotonically increase.
Correspondingly

$$E_2 = \beta \cdot \gamma \sqrt{F_2} \qquad (21)$$

monotonically decrease to $x_0 = \frac{L}{2\gamma}$, and then monotonically increase having a minimum of

$$E_{2,\min im} = E_2\left(x_0 = \frac{L}{2\gamma}\right) = \beta\gamma\, d \qquad (22)$$

For

$$x_0 = -\frac{L}{2\gamma} \qquad (23)$$

$$E_2\left(x_0 = -\frac{L}{2\gamma}\right) = \beta \cdot \gamma \sqrt{d^2 + L^2} \qquad (24)$$

Graphically, the dependence $E_2(x_0)$ looks like in the ***Figure 3***
Expression

$$\begin{cases} E = E_1 + E_2 = \gamma\beta\left(\sqrt{F_2} - \sqrt{F_1}\right) \\ E = \gamma\beta\,\dfrac{F_2 - F_1}{\left(\sqrt{F_2} + \sqrt{F_1}\right)} = \dfrac{\left(\gamma \cdot x_0 - \frac{L}{2}\right)^2 + d^2 - \left(\left(\gamma \cdot x_0 + \frac{L}{2}\right)^2 + d^2\right)}{\sqrt{\left(\gamma \cdot x_0 - \frac{L}{2}\right)^2 + d^2} + \sqrt{\left(\gamma \cdot x_0 + \frac{L}{2}\right)^2 + d^2}} \\ E = \gamma\beta\,\dfrac{-2}{\sqrt{\left(\gamma \cdot x_0 - \frac{L}{2}\right)^2 + d^2} + \sqrt{\left(\gamma \cdot x_0 + \frac{L}{2}\right)^2 + d^2}}\,\gamma \cdot x_0\, L \end{cases} \qquad (25)$$

Monotonically decreases as is shown in the ***Figure 3***
The last relation in (25) can be written as

$$E = \frac{-2\gamma^2 \cdot \beta \cdot L}{\sqrt{\left(\gamma - \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2} + \sqrt{\left(\gamma + \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2}} \cdot \frac{x_0}{|x_0|} \qquad (26)$$

For negative values of x_0, expression becomes

$$E_- = \frac{2\gamma^2 \cdot \beta \cdot L}{\sqrt{\left(\gamma - \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2} + \sqrt{\left(\gamma + \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2}} \qquad (27)$$

If x_0 monotonically increase between $-\infty$ and 0 that is $-\infty < x_0 < 0$, the expression in (28) monotonically decrease

$$E_-(-\infty) = \frac{\gamma^2 \cdot \beta \cdot L}{\sqrt{\gamma^2}} = \gamma \cdot \beta \cdot L \qquad (28)$$

and

$$E_-(0) = 0 \qquad (29)$$

For positive values of x_0, expression (27) is written as

$$E_+ = \frac{-2\gamma^2 \cdot \beta \cdot L}{\sqrt{\left(\gamma - \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2} + \sqrt{\left(\gamma + \frac{L}{2x_0}\right)^2 + \left(\frac{d}{x_0}\right)^2}} \qquad (30)$$

When x_0 monotonically increase between 0 and ∞ that is $0 < x_0 < \infty$, expression (30) monotonically decreases

$$E_+(0) = 0 \qquad (31)$$

and

$$E_+(\infty) = -\gamma \cdot \beta \cdot L \qquad (32)$$

The study of apparent length allows writing that

$$L_i(x_0) = \gamma \cdot L + E \qquad (33)$$

So that, the correct answer to the question ***2b*** is
The apparent length decreases all the time.

Task 3. „Symmetrical image”.

For symmetry reasons, the apparent length on the symmetric picture is the actual length of the moving rod, because the light from the two ends was emitted simultaneously to reach the pinhole at the same time, that is

$$L_{i,simetric} = \frac{L}{\gamma} \qquad (34)^*$$

This is the answer to the question 3a

The apparent endpoint positions are such that

$$x_{i,+} = -x_{i,-} \qquad (35)$$

That is

$$x_{i,+} + x_{i,-} = 0$$

So that, considering (11) , the relation above becomes

$$0 = x_{i,+} + x_{i,-} = 2\gamma^2 \cdot x_0 - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 - \frac{L}{2}\right)^2} \quad (36)$$

In conjunction with the expression of the apparent length of beacons line

$$\frac{L}{\gamma} = x_{i,+} - x_{i,-} = L_i(x_0) = \gamma \cdot L - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} + \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 - \frac{L}{2}\right)^2} \quad (37)$$

Adding the relations above results

$$\begin{cases} \frac{L}{\gamma} = 2\gamma^2 \cdot x_0 + \gamma \cdot L - 2\beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} \\ \frac{2\gamma^2 \cdot x_0 + \gamma \cdot L - \frac{L}{\gamma}}{2\beta \cdot \gamma} = \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} \\ \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} = \frac{x_0 \cdot \gamma}{\beta} + \frac{L\left(\gamma^2 - 1\right)}{2\beta \cdot \gamma^2} = \frac{x_0 \cdot \gamma}{\beta} + \frac{L \cdot \beta}{2} \end{cases} \quad (38)$$

So that

$$\sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} = \frac{x_0 \cdot \gamma}{\beta} + \frac{L \cdot \beta}{2} \quad (39)$$

Subtracting (37) from (38)

$$\sqrt{d^2 + \left(\gamma \cdot x_0 - \frac{L}{2}\right)^2} = \frac{x_0 \cdot \gamma}{\beta} - \frac{L \cdot \beta}{2} \quad (40)$$

One may obtain the position of middle beacon.
From (40) results

$$\begin{cases} d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2 = \left(\frac{x_0 \cdot \gamma}{\beta} + \frac{L \cdot \beta}{2}\right)^2 \\ d^2 + \gamma^2 \cdot x_0^2 + \frac{L^2}{4} + \gamma \cdot x_0 \cdot L = \frac{x_o^2 \cdot \gamma^2}{\beta^2} + \frac{L^2 \cdot \beta^2}{4} + \gamma \cdot x_0 \cdot L \\ \gamma^2 \cdot x_0^2 \cdot \left(1 - \frac{1}{\beta^2}\right) = \frac{L^2}{4}\left(\beta^2 - 1\right) - d^2 \\ \frac{1}{1 - \beta^2} \cdot x_0^2 \cdot \left(1 - \frac{1}{\beta^2}\right) = \frac{L^2}{4}\left(\beta^2 - 1\right) - d^2 \end{cases} \quad (41)$$

So that

$$x_0 = \beta \sqrt{\frac{L^2}{4\gamma^2} + d^2} \quad (42)^*$$

*The relation (42) represents the answer for **3.b.***

An alternative method to determine the position of the middle beacon uses the hypothesis of symmetric ends – that is relation (37) written as

$$2\gamma^2 \cdot x_0 - \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} = \beta \cdot \gamma \sqrt{d^2 + \left(\gamma \cdot x_0 - \frac{L}{2}\right)^2} \qquad (43)$$

Squaring,

$$\begin{cases} 4\gamma^4 \cdot x_0^2 + 2\beta^2 \cdot \gamma^3 \cdot x_0 \cdot L = 4\beta \cdot \gamma^3 \cdot x_0 \cdot \sqrt{d^2 + \left(\gamma \cdot x_0 + \frac{L}{2}\right)^2} \\ 4\gamma^2 \cdot x_0^4 \left(1 - \beta^2\right) + \beta^2 \cdot x_0^2 \cdot L^2 \left(\beta^2 - 1\right) - 4\beta^2 \cdot x_0^2 \cdot d^2 = 0 \\ 4 \cdot x_0^4 - \frac{\beta^2 \cdot x_0^2 \cdot L^2}{\gamma^2} - 4\beta^2 \cdot x_0^2 \cdot d^2 = 0 \end{cases} \qquad (44)$$

From last relation results again (43).

The position of middle beacon (8)

$$\begin{cases} x_{i,0} = \gamma\left(\gamma \cdot x_0 - \beta \cdot \sqrt{d^2 + \gamma^2 \cdot x_0^2}\right) \\ x_{i,0} = \gamma\left(\gamma \cdot \beta \sqrt{d^2 + \left(\frac{L}{2\gamma}\right)^2} - \beta \cdot \sqrt{d^2 + \gamma^2 \cdot \beta^2 \left(d^2 + \left(\frac{L}{2\gamma}\right)^2\right)}\right) \end{cases} \qquad (45)$$

The image of the middle of the rod on the symmetric picture is, therefore, located at

$$x_{i,0} = \gamma \cdot \beta \left(\sqrt{(\gamma \cdot d)^2 + \left(\frac{L}{2}\right)^2} - \sqrt{(d \cdot \gamma)^2 + \left(\frac{L \cdot \beta}{2}\right)^2}\right) \qquad (46)$$

The image of the front end of beacons line is $x_{i,+}$. The distance between the image of the end and the image of middle beacon is

$$\ell = x_{i,+} - x_{i,0} = \frac{L}{2\gamma} - x_{i,0} = \frac{L}{2\gamma} - \gamma \cdot \beta \left(\sqrt{(\gamma \cdot d)^2 + \left(\frac{L}{2}\right)^2} - \sqrt{(d \cdot \gamma)^2 + \left(\frac{L \cdot \beta}{2}\right)^2}\right) \qquad (47)^*$$

*(The expression above is the answer to the question **3.c**)*

Task 4. „Images of SS Enterprise being far away, approaching and receding."

The relations (29) and (34) allow to write for an image taken for SS being far away and coming the expression of length as

$$\begin{cases} L_{i,apropiere} = L_i(x_0 \to -\infty) = \gamma \cdot L + E_- = \gamma \cdot L + \gamma \cdot \beta \cdot L \\ L_{i,apropiere} = \gamma \cdot L \cdot (1+\beta) = L \dfrac{1+\beta}{\sqrt{1-\beta^2}} \\ L_{i,apropiere} = L\sqrt{\dfrac{1+\beta}{1-\beta}} \end{cases} \tag{48}$$

Analogous, for the length of SS Enterprise being far away and moving away the relations (33) şi (34) allow to write

$$\begin{cases} L_{i,indepartare} = L_i(x_0 \to \infty) = \gamma \cdot L + E_+ = \gamma \cdot L - \gamma \cdot \beta \cdot L \\ L_{i,indepartare} = \gamma \cdot L \cdot (1-\beta) = L \dfrac{1-\beta}{\sqrt{1-\beta^2}} \\ L_{i,indepartare} = L\sqrt{\dfrac{1-\beta}{1+\beta}} \end{cases} \tag{49}$$

Because –

$$L_{i,apropiere} = L\sqrt{\frac{1+\beta}{1-\beta}} > L_{i,indepartare} = L\sqrt{\frac{1-\beta}{1+\beta}} \tag{50}$$

The correct answer to the question 4a is
***b.** The apparent length is $600\ m$ on the image of coming ship and $200\ m$ on the image of ship receding.*
The ratio of length is

$$\frac{L_{i,apropiere}}{L_{i,indepartare}} = \frac{L\sqrt{\dfrac{1+\beta}{1-\beta}}}{L\sqrt{\dfrac{1-\beta}{1+\beta}}} = \frac{1+\beta}{1-\beta} = \frac{3}{1}. \tag{51}$$

So that

$$\begin{cases} 1+\beta = 3 - 3\beta \\ \beta = \dfrac{1}{2} \end{cases} \tag{52}$$

Consequently

$$v = \frac{c}{2} \tag{53}$$

*(The relation (53) is the answer to the question**4.b)***
Considering the expression of the length of beacons line for the coming ship,

$$600\ m = L_{i,apropiere} = L\sqrt{\frac{1+1/2}{1-1/2}} = L\sqrt{3} \tag{54}$$

Results that

$$L = 200\sqrt{3}\, m \cong 346\ m \qquad (55)$$

(The value above is the answer to the question 4.c)

Because γ for $\beta = \frac{1}{2}$ is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}} \qquad (56)$$

The length of symmetrical image is

$$\begin{cases} L_i = \frac{L}{\gamma} = 200\sqrt{3}\frac{\sqrt{3}}{2} \\ L_i = 300\ m \end{cases} \qquad (57)$$

The value above is the answer to the question 4.d.